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Lipschitz Continuity and Strong Unicity in G. Freud's Work

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DEDICATED TO THE MEMORY OF GÉZA FREUD

The purpose of this paper is to illustrate the significance of G. Freud's work on Lipschitz continuity of the operator of best uniform approximation and the impact of his investigations on strong unicity and strong unicity constants.

Let $C[a, b]$ be the collection of all continuous real-valued functions on the interval $[a, b]$ endowed with the uniform norm $\|\cdot\|$, and V_n a subspace of $C[a, b]$ of dimension n. For each $f \in C[a, b]$ let

$$P_{V_n}(f) := \{v \in V_n \mid \|f - v\| = \rho_n(f)\} \tag{1}$$

be the set of all best uniform approximations of f from V_n, where

$$\rho_n(f) := \inf_{v \in V_n} \|f - v\|. \tag{2}$$

If V_n is a Haar subspace on $[a, b]$, then there exists for every $f \in C[a, b]$ exactly one element v_f in $P_{V_n}(f)$, and we define the operator

$$T_n: \quad C[a, b] \to V_n \tag{3}$$

of best uniform approximation by $T_n(f) := v_f$. Moreover, for each $\varepsilon \geqslant 0$ define

$$P^f_{V_n}(\varepsilon) := \{v \in V_n \mid \|f - v\| \leqslant (1 + \varepsilon)\, \rho_n(f)\}. \tag{4}$$

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Then for $V_n = \prod_{n-1}$, the collection of all real polynomials of degree $\leqslant n-1$, C. de la Vallée Poussin [16] proved that the set-valued mapping

$$P^f_{V_n}: \quad [0, \infty) \to 2^{V_n}$$

is continuous at the point 0. Consequently, if $V_n = \prod_{n-1}$, the mapping T_n is continuous at every point $f \in C[a, b]$.

Freud [5] seems to have been the first to give a quantitative version of this theorem of de la Vallée Poussin, namely to investigate the pointwise Lipschitz continuity of the operator T_n.

THEOREM 1 (Freud [5]). *If V_n is a Haar subspace of $C[a, b]$, then there exists a constant $A > 0$, A depending on f and V_n, such that*

$$\|T_n(f) - T_n(g)\| \leqslant A \|f - g\| \qquad \text{for all } g \in C[a, b]. \tag{5}$$

It is very interesting to look into the original proof of this theorem, especially because nowadays Theorem 1 is usually proved by using the strong unicity theorem (cf. [3]): First note that

$$|\rho_n(f) - \rho_n(g)| \leqslant \|f - g\|. \tag{6}$$

Moreover, there exist $n+1$ points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that

$$\varepsilon(-1)^i (f - T_n(f))(x_i) = \rho_n(f) \tag{7}$$

($\varepsilon = +1$ or $\varepsilon = -1$) for $i = 0, \ldots, n$.
Then, for $0 \leqslant i \leqslant n$ and any $v \in V_n$,

$$\begin{aligned} &\varepsilon(-1)^i (T_n(f) - v)(x_i) \\ &\quad = \varepsilon(-1)^i [(T_n(f) - f)(x_i) + (f - v)(x_i)] \\ &\quad \leqslant -\rho_n(f) + \|f - v\|. \end{aligned} \tag{8}$$

If $v = T_n(g)$, $g \in C[a, b]$, and $\|f - g\| \leqslant \delta$, then the inequalities (6) and (8) yield

$$\varepsilon(-1)^i (T_n(f) - v)(x_i) \leqslant 2\delta. \tag{9}$$

Omitting the rth point ($0 \leqslant r \leqslant n$), Freud considered now the point set

$$x_0 < x_1 < \cdots < x_{r-1} < x_{r+1} < \cdots x_n \tag{10}$$

and the fundamental polynomials $L_{i,r}(x) \in V_n$, $0 \leqslant i \leqslant n$, $i \neq r$, of Lagrange interpolation corresponding to the set (10):

$$L_{i,r}(x) = \delta_{i,j} \qquad \text{for } 0 < i, j \leqslant n;\ i, j \neq r.$$

Further, let $x_{-1} := a$, $x_{n+1} := b$ and define

$$\lambda_r := \max_{x_{r-1} \leqslant x \leqslant x_{r+1}} \sum_{\substack{i=0 \\ i \neq r}}^{n} |L_{i,r}(x)| \qquad \text{for } 0 \leqslant r \leqslant n, \tag{11}$$

$$\lambda_{-1} := \max_{a \leqslant x \leqslant x_0} \sum_{i=0}^{n-1} |L_{i,n}(x)|, \tag{12}$$

$$\lambda_{n+1} := \max_{x_n \leqslant x \leqslant b} \sum_{i=1}^{n} |L_{i,0}(x)|, \tag{13}$$

$$\lambda := \max_{-1 \leqslant i \leqslant n+1} \lambda_i. \tag{14}$$

Since

$$\operatorname{sgn} L_{i,r}(x) = (-1)^{i-r+1} \qquad \text{for } x \in (x_{r-1}, x_{r+1}), \tag{15}$$

we obtain for the function

$$\tilde{v}(x) := \varepsilon(T_n(f) - T_n(g))\,(x) \in V_n$$

and $x \in [x_{r-1}, x_{r+1}]$, $0 \leqslant r \leqslant n$,

$$\begin{aligned} (-1)^{r+1}\,\tilde{v}(x) &= (-1)^{r+1} \sum_{\substack{i=0 \\ i \neq r}}^{n} \tilde{v}(x_i)\, L_{i,r}(x) \\ &= \sum_{\substack{i=0 \\ i \neq r}}^{n} \varepsilon(-1)^i\,(T_n(f) - T_n(g))\,(x_i)|L_{i,r}(x)| \\ &\leqslant 2\delta\lambda_r. \end{aligned} \tag{16}$$

Combining (16) with the corresponding inequality for the index $r+1$, we get $|\tilde{v}(x)| \leqslant 2\delta\lambda$ for $x \in [x_r, x_{r+1}]$ and each $0 \leqslant r \leqslant n-1$ or

$$|\tilde{v}(x)| \leqslant 2\delta\lambda \qquad \text{for } x \in [x_0, x_n]. \tag{17}$$

Since

$$\operatorname{sgn} L_{i,n}(x) = (-1)^i \qquad \text{for } x \in [a, x_0),\ i = 0, \ldots, n-1, \tag{18}$$

we get

$$\tilde{v}(x) = \sum_{i=0}^{n-1} \tilde{v}(x_i)\, L_{i,n}(x) \leqslant 2\delta\lambda_{-1}; \tag{19}$$

the same arguments lead to

$$\operatorname{sgn} L_{i,0}(x) = (-1)^{n-i} \qquad \text{for } x \in (x_n, b],\ i = 1, \ldots, n, \tag{20}$$

and therefore

$$(-1)^n \tilde{v}(x) = (-1)^n \sum_{i=1}^{n} \tilde{v}(x_i)\, L_{i,0}(x) \leqslant 2\delta\lambda_{n+1}. \tag{21}$$

Combining (19) and (21) with the inequality (16) for $r = 0$ and $r = n$, we obtain finally together with (17),

$$\|\tilde{v}\| \leqslant 2\delta\lambda$$

or

$$\|T_n(f) - T_n(g)\| \leqslant 2\lambda \|f - g\|. \quad \blacksquare \tag{22}$$

Since the arguments in the proof of Freud do not depend on $T_n(g) \in V_n$, but may be applied to any $v \in V_n$, Freud has actually proved that

$$\|T_n(f) - v\| \leqslant \lambda \max_{0 \leqslant i \leqslant n} [\varepsilon(-1)^i (T_n(f) - v)(x_i)]$$

or, using (8),

$$\|f - v\| \geqslant \|f - T_n(f)\| + \lambda^{-1} \|T_n(f) - v\|. \tag{23}$$

This means that the best uniform approximation $T_n(f)$ is *strongly unique* for every $f \in C[a, b]$, a result extended by Newman and Shapiro [11] for the approximation of $f \in C(X)$ by a Chebyshev system, where $C(X)$ is the Banach space of continuous functions on a compact Hausdorff space X. Moreover Freud has given a lower bound for the *strong unicity constant* which is the largest constant γ, denoted by $\gamma_n(f)$, such that the inequality

$$\|f - v\| \geqslant \|f - T_n(f)\| + \gamma \|T_n(f) - v\| \tag{24}$$

holds. Namely, from (23), we know that

$$\gamma_n(f) \geqslant \lambda^{-1}, \tag{25}$$

where λ is defined by (14).

To connect the lower bound (25) with the lower bound given by Cline [4], let us consider again the point set $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ satisfying (7). We define the function $v_r \in V_n$ such that

$$v_r(x_i) = \varepsilon(-1)^i = \operatorname{sgn}(f - T_n(f))(x_i) \tag{26}$$

for $i=0,\ldots,n$, $i\neq r$. Then the definitions (11)–(14) and the properties (15), (18), (20) yield

$$K := \max_{0\leqslant r\leqslant n} \|v_r\| \geqslant \lambda. \tag{27}$$

So $1/K$ is again a lower bound for the strong unicity constant, a result proved by Cline [4] for the approximation of $f\in C(X)$ by a Chebyshev system.

At first glance, by (27), the lower bound of Freud in (25) seems to be stronger than the lower bound $\gamma_n(f)\geqslant 1/K$ of Cline. But a closer look shows that $K=\lambda$ and both lower bounds are the same.

In the last ten years quite a number of papers were concerned with estimations for strong unicity constants, expecially in the case $V_n=\prod_{n-1}$: Poreda [13] raised the question to describe the asymptotic behaviour of $\gamma_n(f)$ for fixed f and $n\to\infty$. Later, Henry and Roulier [6] conjectured that

$$\varliminf_{n\to\infty} \gamma_n(f)=0 \qquad \text{or} \qquad f \quad \text{is a polynomial} \tag{28}$$

if $V_n=\prod_{n-1}$, $n=1, 2,\ldots$. This problem is still open, it is only solved for special classes of functions [2, 6–10, 14]. Finally we want to mention that recently Kroo [10] used the lower bound in (21) for obtaining lower asymptotic estimates for $\gamma_n(f)$, $n\to\infty$.

Another interesting aspect of Freud's proof of Theorem 1 is reflected in the case when the subspace V_n *does not satisfy the Haar condition*:

Let V_n be a weak Chebyshev subspace satisfying the condition (N), i.e., there exists a point $z\in[a, b]$ such that each $v\in V_n$, $v\neq 0$, has at most $n-1$ zeros in $[a, b]\setminus\{z\}$. Then for every $f\in C[a, b]$ there exists a best uniform approximation $v_f\in V_n$, characterized by an alternating point set

$$a\leqslant x_0<x_1<\cdots<x_n\leqslant b, \tag{29}$$

i.e.,

$$\varepsilon(-1)^i\,(f-v_f)\,(x_i)=\rho_n(f) \tag{30}$$

($\varepsilon=+1$ or $\varepsilon=-1$). v_f is called *alternation element* of $P_{V_n}(f)$ and v_f is unique (Nürnberger, Sommer [12], Sommer, Strauß [15]).

THEOREM 2. *Let V_n be a weak Chebyshev subspace of $C[a, b]$ with property (N). Then the selection $s\colon C[a, b]\to V_n$, which associates to every $f\in C[a, b]$ the unique alternation element $s(f)=v_f$ of $P_{V_n}(f)$, is a pointwise Lipschitz continuous mapping.*

Proof. Assume, s is not Lipschitz continuous at $f \in C[a, b]$. Then $f \notin V_n$ and there exists a sequence $\{f_m\}$ in $C[a, b]$ such that $f_m \to f$ and

$$\|s(f) - s(f_m)\| > m\|f - f_m\|. \tag{31}$$

$f - s(f_m)$ has an alternating point set

$$a \leqslant x_0^{(m)} < x_1^{(m)} < \cdots < x_n^{(m)} \leqslant b, \tag{32}$$

where $\varepsilon_m(-1)^i (f_m - s(f_m))(x_i^{(m)}) = \rho_n(f_m)$ for $i = 0, ..., n$ ($\varepsilon_m = +1$ or $\varepsilon_m = -1$). We may assume that $x_i^{(m)} \to x_i$ for $m \to \infty$ ($i = 0, ..., n$), and $\varepsilon_m = \varepsilon$ ($\varepsilon = +1$ or $\varepsilon = -1$) is fixed for all m. Then

$$a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b, \tag{33}$$

where $\varepsilon(-1)^i (f - s(f))(x_i) = \rho_n(f)$ for $i = 0, ..., n$. Now we omit a fixed point x_r in (33), where x_r is chosen as the point z in the condition (N), if $z \in \{x_0, x_1, ..., x_n\}$. Following Freud's proof of Theorem 1, we obtain, as in (16), for $x \in [x_{r-1}, x_{r+1}]$ that

$$(-1)^{r+1} \varepsilon(s(f) - s(f_m))(x) \leqslant 2\lambda_r \|f - f_m\|, \tag{34}$$

where $x_{-1} := a$, $x_{n+1} := b$ and λ_r is defined by (11). Analogously we have for $x \in [x_{r-1}^{(m)}, x_{r+1}^{(m)}]$ the inequality

$$(-1)^{r+1} \varepsilon(s(f_m) - s(f))(x) \leqslant 2\lambda_r^{(m)} \|f - f_m\|, \tag{29}$$

where again $x_{-1}^{(m)} := a$, $x_{n+1}^{(m)} := b$,

$$\lambda_r^{(m)} := \max_{x_{r-1}^{(m)} \leqslant x \leqslant x_{r+1}^{(m)}} \sum_{\substack{i=0 \\ i \neq r}}^{n} |L_{i,r}^{(m)}(x)| \tag{36}$$

and the functions $L_{i,r}^{(m)}(x) \in V_n$ are the fundamental polynomials of Lagrange interpolation corresponding to the set

$$x_0^{(m)} < x_1^{(m)} < \cdots < x_{r-1}^{(m)} < x_{r+1}^{(m)} < \cdots < x_n^{(m)}.$$

Since $L_{i,r}^{(m)} \to L_{i,r}$, and therefore $\lambda_r^{(m)} \to \lambda_r$ for $m \to \infty$, there exists a constant $\gamma_1 > 0$ such that

$$|(s(f) - s(f_m))(x)| \leqslant \gamma_1 \|f - f_m\| \tag{37}$$

for all $x \in [x_{r-1} + \delta, x_{r+1} - \delta]$ with $\delta := (x_{r+1} - x_{r-1})/4$. Since for $v \in V_n$ the uniform norm on $[a, b]$ is equivalent to the uniform norm on the interval $[x_{r-1} + \delta, x_{r+1} - \delta]$, inequality (37) contradicts (31).

A generalization of Theorem 2 for generalized spline spaces V_n is proved by Blatt, Nürnberger, Sommer [1], again using as an essential tool to the idea of Freud, outlined in the above proof.

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